Springbok: The Physics of Jumping

Robert J. Dufresne, William J. Gerace, and William J. Leonard

A simple spring-loaded toy that jumps up off the table when compressed and released offers the physics teacher a rich context for exploring a wide range of physics concepts and principles, and it possesses a number of features that give it broad instructional value.

In the following, we refer to this system (basically two masses and a spring, illustrated in Fig. 1) as a *springbok*, named for the South African gazelle that is noted for its grace and delightful habit of suddenly springing into the air. A jumping springbok is easy to make and engaging to study, amenable to both conceptual and quantitative analyses at a variety of levels.

The springbok system raises intriguing questions: Why does it jump? Under what circumstances will it leave the table? Will a springbok jump to the same height on the Moon as on Earth? If the masses on the two ends are unequal, for which orientation of the masses will the springbok jump higher? From introductory to graduate instructional situations, teachers can choose to approach the physics of jumping either empirically — by building a springbok and observing its behavior — or theoretically — by creating a quantitative model and analyzing its implications. However used, the springbok will provide students with an enjoyable learning experience.

We begin here with a quantitative analysis based on a simple (Hooke's law) model, follow that with a conceptual analysis of jumping, and then give a sampling of further ways to use a springbok. For a more detailed presentation, see Ref. 1.

Quantitative Analysis

An Elementary Model. The model we will use to analyze a springbok contains three simplifying assumptions: (1) the spring connecting the two masses obeys Hooke's law, (2) the mass of the spring is negligibly small, and (3) all dissipative effects can be ignored, including energy loss as the spring oscillates. Even with these assumptions, analyzing a springbok is complicated because it consists of two masses that interact with, and move relative to, one another. Nevertheless, two tactics will permit a straightforward solution within this model. First, divide the motion of the springbok into two distinct phases: phase 1 — while the springbok is in contact with the table, and phase 2 — after the springbok has left the table. Then focus the analysis on the center of mass of the springbok.

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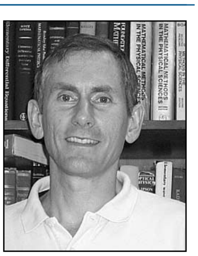

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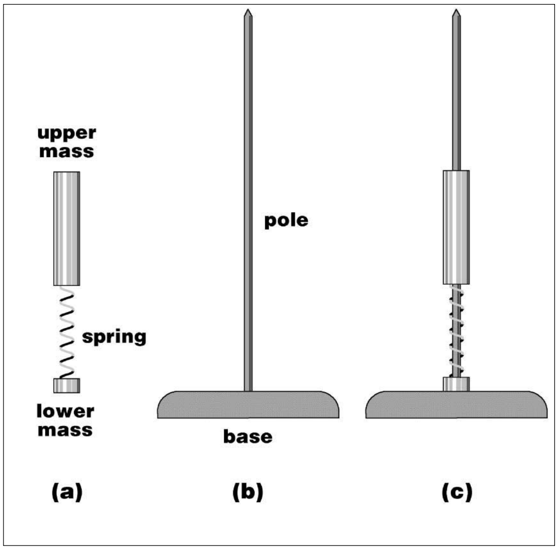


Fig. 1. (a) Diagram of springbok showing its components. (b) A pole and base are used to maintain stable vertical motion. (c) Holes are drilled through the centers of the upper and lower masses so that the springbok may be placed over the pole.

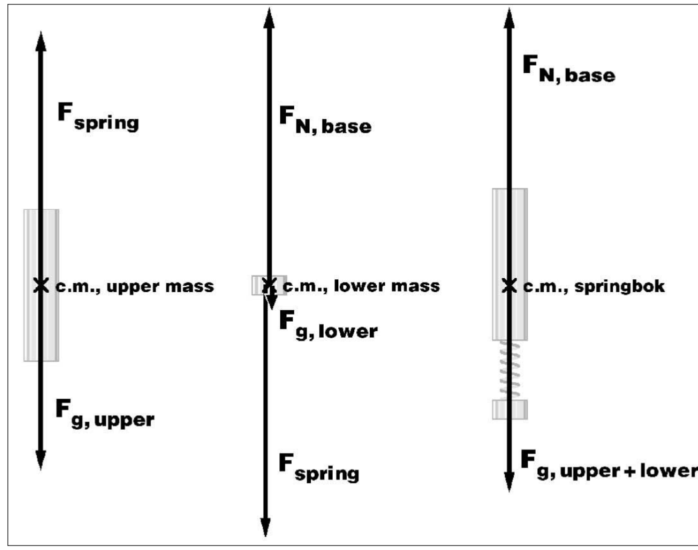


Fig. 2. Free-body diagrams for upper mass, lower mass, and springbok systems. Assumes springbok is in contact with table and that its center of mass is accelerating upward.

The Equations of Motion. Free-body diagrams for the two masses are shown in Fig. 2. Applying Newton's second law leads to the following equations of motion during phase 1:

$$m_u \ddot{y}_u = -k(y_u - y_l - L) - m_u g \tag{1a}$$

and

$$m_l \ddot{y}_l = k(y_u - y_l - L) - m_l g + F_N. \tag{1b}$$

In these equations, m_u is the upper mass, m_l the lower mass, k the spring constant, L the natural length of the spring, g the gravitational field strength, and F_N the normal force exerted on the lower mass by the table. These two equations are written in terms of the upper and lower-mass coordinates y_u and y_l. This form is most useful for analyzing phase 1 (i.e., while the lower mass remains in contact with the table).

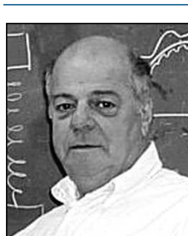
For phase 2, the equations of motion are the same as for phase 1, except that the normal force on the lower mass is zero. The equations of motion are easier to solve when written in terms of the total and reduced masses ($M = m_u + m_l$ and $\mu = m_u m_l / M$), and the center-of-mass (c.m.) and relative coordinates [$Y = (m_u y_u + m_l y_l)/M$ and $y = y_u - y_l$]. The resulting equations are:

$$M\ddot{Y} = -Mg \tag{2a}$$

and

$$\mu\ddot{y} = -k(y - L). \tag{2b}$$

Solving the Equations of Motion. During phase 1, the lower-mass coordinate is constant in time. If we take its position to be zero ($y_l = 0$), Eq. (1b) becomes trivial (i.e., $\ddot{y}_l = 0$). Equation (1a) becomes identical to the equation of motion for an object attached to a vertical Hooke's-law spring, and is easily solved for the upper-mass coordinate (y_u) as a function of time. [Note that once y_u is determined, we can obtain the normal force



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Phase 1 solution:

$$y_u(t) = y_{eq} - y_0 \cos \alpha t \qquad y(t) = y_u(t)$$

$$y_l(t) = 0 \qquad Y(t) = \frac{m_u}{M} y_u(t)$$

where:

$$y_{eq} = L - \frac{m_u g}{k}$$

$$y_0 = D - \frac{m_u g}{k}$$

$$\alpha = \sqrt{\frac{k}{m_u}}$$

$D =$ amount the spring is compressed

a.

Phase 2 solution:

$$y(t) = L + \frac{m_l g}{k} \cos \beta t' + \frac{v_0}{\beta} \sin \beta t'$$

$$Y(t) = Y_0 + V_0 t' - \tfrac{1}{2} g (t')^2$$

where:

$$v_0 = \frac{\alpha}{k}\left[\left(kD - m_u g\right)^2 - \left(Mg\right)^2\right]^{1/2}$$

$$\beta = \sqrt{\frac{k}{\mu}}$$

$$Y_0 = \frac{m_u}{M}\left(L + \frac{m_l g}{k}\right)$$

$$V_0 = \frac{m_u}{M} v_0$$

$$t' = t - \frac{1}{\alpha} \cos^{-1}\left[\frac{Mg}{m_u g - kD}\right]$$

b.

Fig. 3. Solutions to equations of motion for (a) Phase 1: interval while springbok is in contact with table; and (b) Phase 2: interval after springbok leaves table. Time t′ = 0 corresponds to time at which springbok leaves table.

as a function of time by substituting y_u into Eq. (1b).] The solution for phase 1 is presented in Fig. 3a.

During phase 2, the springbok is no longer in contact with the table and the normal force is zero. When this is the case, the equations of motion are uncoupled if they are written in terms of the center-of-mass and relative coordinates. Equation (2a), which describes the center-of-mass coordinate, is equivalent to the equation of motion for an object experiencing a constant gravitational force. Equation (2b), which describes the relative coordinate, is equivalent to the equation of motion for an object attached to a Hooke's-law spring. Both equations are straightforward to solve. The solution for phase 2 is presented in Fig. 3b.

Obtaining the general solution for each phase is not particularly difficult. However, finding the specific solution for a given set of initial conditions involves a bit of algebra, which results from matching the phase 1 and phase 2 general solutions at the instant the lower mass leaves the table. This occurs when the spring is slightly *stretched*, and the spring force pulling up on the lower mass is equal to its weight ($ky_u = m_l g$), a condition that determines the position of the upper mass. The upper-mass position can then be used to find the speed of the upper mass (e.g., by using conservation of energy) and the time that the lower mass leaves the table [e.g., by inverting

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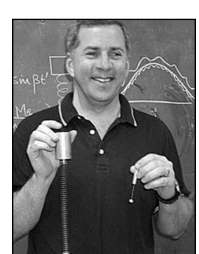

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the solution to Eq. (1a)]. How the various constants in the general solution are related to the initial conditions is indicated in Fig. 3.

A graphical representation of the solution for a particular set of parameters and initial conditions is provided in Fig. 4. The model predicts that the center of mass of the springbok will reach a larger maximum height when the larger mass is on top. It also predicts that the springbok will stay in the air much longer when the larger mass is on top. Numerically, the results are that the center of mass reaches a maximum height of about 46 cm at around 0.3 seconds after being released when the larger mass is on top, and it reaches a maximum height of only about 6 cm at around 0.12 seconds when the smaller mass is on top.

The actual motion of a springbok is shown in Fig. 5. Thirty frames per second were taken using a digital camera, and a selection of them are shown. When the smaller mass is on top, the center of mass reaches a maximum height of about 6.5 cm at around 0.1 seconds after being released (i.e., about the fourth frame), and when the larger mass is on top, the center of mass reaches a maximum height of about 45 cm at around 0.3 seconds (i.e., the 11th frame). These observed values match up very well with the predicted values. The solutions to the equations of motion and the actual motion of a real springbok both show the same result: When the upper mass is heavier than the lower mass, the springbok is in the air longer, and its maximum height is larger.

Two caveats should be mentioned. First, in writing down the solution, we have assumed that the springbok loses contact with the table (i.e., the normal force is zero at some time). The springbok will leave the table if the upper mass stretches the spring to a point where the spring force on the lower mass is equal to its weight. This condition will be met only if the spring is initially compressed beyond some minimal amount determined by the size of the spring constant, the values of the two masses, and the strength of the gravitational field. Second, the phase 2 solution remains valid only while the position of the lower mass is greater than zero.

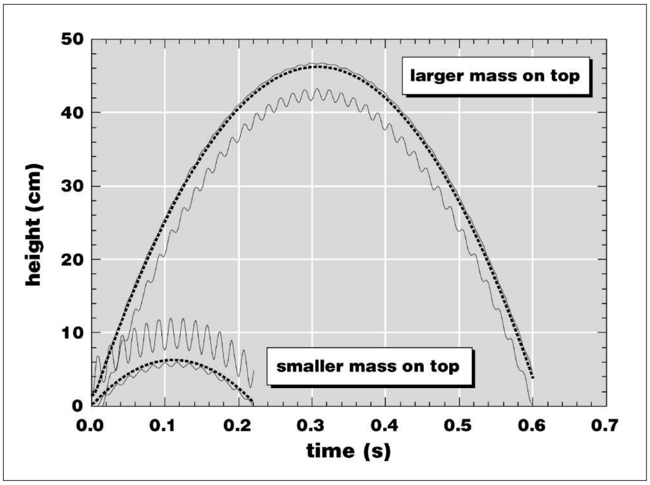

Fig. 4. Plots of upper mass, center of mass, and lower mass height vs time for two orientations of springbok: larger mass on top and smaller mass on top. Parameters correspond to the springbok shown in Fig. 5.

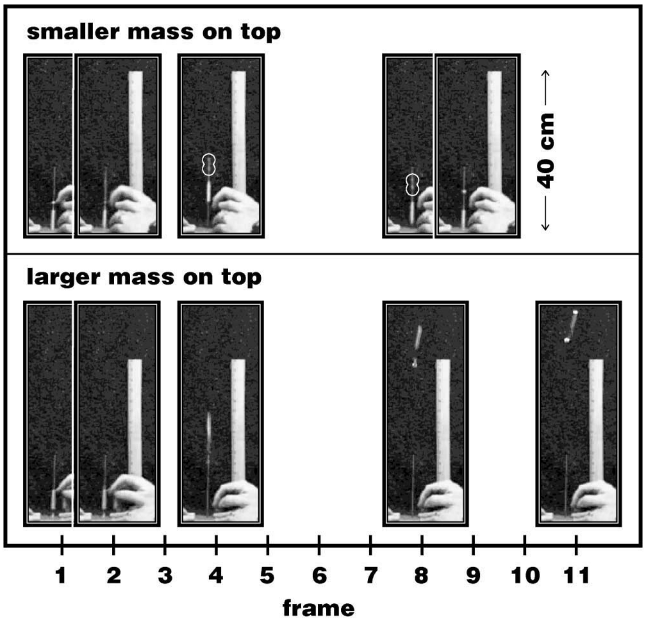

Fig. 5. Frames from digital video showing motion of a springbok for two orientations: larger mass on top and smaller mass on top.

Finding the Maximum Height Without Solving the Equations of Motion. For those not interested in the details of the motion, but only in the maximum height of the center of mass, a solution can be found without solving the equations of motion. In other words, an algebraic expression for the maximum height is possible without solving any differential equations.

Initially, the spring is compressed a distance D. The lower mass maintains contact with the table until the spring force on the lower mass balances its weight, which occurs when the spring is stretched a distance d from its relaxed state:

$$d = \frac{m_l g}{k}. \tag{3}$$

During phase 1, the upper mass moves up a total distance $D + d$, producing a change in height of the center of mass:[2]

$$\Delta h_{\mathrm{cm}}(\text{Phase 1}) = \frac{m_u}{m_u + m_l}(D + d)\,. \tag{4}$$

Once the springbok loses contact with the table, the only external force on the toy is the gravitational force exerted by Earth, and so, the acceleration of the toy's center of mass is constant and equal to g. For constant acceleration, the maximum change in height of the center of mass is easily related to the velocity of the center of mass (V_0) at the moment the springbok loses contact with the table:

$$\Delta h_{\mathrm{cm}}(\text{Phase 2}) = \frac{V_0^2}{2g}\,. \tag{5}$$

At this instant, only the upper mass is moving. In terms of the upper-mass velocity (v_0), the center of mass velocity is

$$V_0 = \frac{m_u}{m_u + m_l}\, v_0\,. \tag{6}$$

If dissipative effects can be ignored, the velocity of the upper mass (at the moment the toy loses contact with the table) can be found using conservation of energy. There are three contributions to the energy: (1) elastic potential energy in the stretched spring, (2) gravitational potential energy in the Earth-upper mass system, and (3) kinetic energy due to the motion of the upper mass:

$$\tfrac{1}{2}kD^2 = \tfrac{1}{2}kd^2 + m_u g(D + d) + \tfrac{1}{2}m_u v_0^2\,. \tag{7}$$

By working backward, Eqs. (5), (6), and (7) can be used to find the maximum change in height of the springbok's center of mass for phase 2. This result can be added to the change in height for phase 1 [Eq. (4)] to obtain a single expression — in terms of d, D, m_u, m_l, and k — for the maximum change in height of the center of mass for phases 1 and 2 combined.

$$\Delta h_{\mathrm{cm}}(\text{Phase 1 + Phase 2})$$

$$= \left(\frac{m_u}{m_u + m_l}\right)\left(1 + \frac{d}{D}\right)^2\left[\frac{\tfrac{1}{2}kD^2}{(m_u + m_l)g}\right] \tag{8}$$

With this result students can investigate the question about which orientation — the larger mass on top or the smaller mass on top — produces the greater change in the height of the springbok's center of mass. For a sufficiently stiff spring (i.e., $kD >> Mg$), the springbok will leave the table after stretching a small distance $d << D$. In this limit, the change in height of the center of mass is proportional to the upper mass, and therefore is greater when the larger mass is on top.

Conceptual Analysis

Teachers might want to address some of the following conceptual issues before having their students attempt a formal solution such as presented in the Quantitative Analysis section.

Force and Acceleration. Initially, the springbok's center of mass accelerates upward, away from the table. According to Newton's laws, this is possible only if there is an upward, net external force exerted on the springbok. The only external force that points in the upward direction is the normal force. To realize a net upward force, the normal force must exceed the weight of the springbok. Some students may believe that the normal force is *always* greater than the weight of the springbok while it is in contact with the table, when in fact it ranges from being larger than the weight (when the spring is maximally compressed) to being equal to zero (when the springbok loses contact with the table). Other students may think that the normal force is always equal to the springbok's weight.

Students may have difficulty identifying the force that causes the springbok to jump off the table. For example, some may think that the spring provides the net upward force on the springbok. Drawing and analyzing free-body diagrams for different systems (e.g., the upper mass, lower mass, and springbok) and different times (e.g., the point of release, the moment the springbok leaves the table, and while the springbok is in the air) should help students clarify some of these points.

Work and Kinetic Energy. Initially the springbok is at rest, but by the time it leaves the table it has acquired kinetic energy. The work-kinetic energy theorem implies that a positive amount of work is done on the springbok during phase 1. Students may have difficulty identifying the force that does the positive work on the springbok. In the table's frame of reference, the normal force exerted by the table does no work on the springbok. This is because the table and the lower

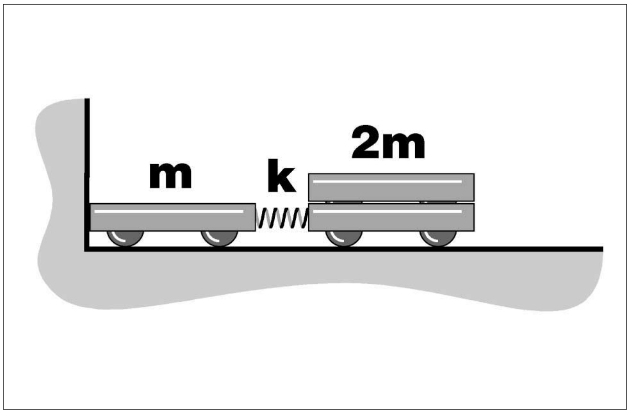

Fig. 6. Two carts connected to a compressed spring and launched from a wall — the horizontal "equivalent" of the springbok system.

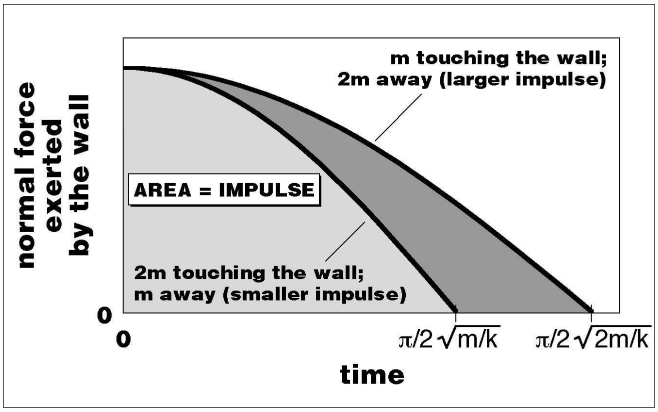

Fig. 7. Normal force vs time exerted by the wall on the two-cart system (Fig. 6) for two different orientations.

mass remain fixed during phase 1. The gravitational force does negative work on the upper mass as it moves upward. So, which force does positive work?

A springbok is a simple example of a system for which the internal forces do net work on the system. Specifically, the spring does net positive work on the springbok as it relaxes from its initially compressed state. Interestingly, although the spring does net work, it does not contribute to the net force on the springbok. The opposite is true for the table, which does exert a net upward force on the springbok, but does no work.

Conservation of Energy. Students sometimes have a limited concept of conservation of energy based on their study of particle mechanics. For a single block launched from a spring, for example, the maximum height of the block is determined by the mass of the block and the distance the spring is initially compressed. This follows easily from conservation of energy, which requires that the elastic potential energy, initially stored in the spring, be equal to the increase in gravitational potential energy as the block rises to its maximum height. Students might reason in an analogous way about springbok and conclude that the orientation of the springbok does not matter — after all, neither the amount of initial elastic potential energy nor the total mass of the springbok depend upon which mass is on top. Some students will be unaware that the springbok has kinetic energy at the point where the center of mass is at its maximum height. But even those who are aware may incorrectly believe that the potential energy initially stored in the spring is equal to the gain in gravitational potential energy.

What students might not realize is that for a multiparticle system such as this one, the energy in the system gets divided among different degrees of freedom in a way that depends upon the details of the motion. For a springbok, the initial elastic potential energy is transformed into gravitational potential energy, kinetic energy associated with the motion of the center of mass, and kinetic energy associated with the motion of the masses relative to the center of mass. At the maximum height, the kinetic energy associated with the motion of the center of mass is zero, but the kinetic energy associated with the motion of the two masses relative to the center of mass is not. Further, at the maximum height there will usually remain some elastic potential energy — the spring will not be at its natural length. In deciding how to make the springbok jump the highest, one is effectively determining how to maximize the transfer of the initial elastic potential energy into translational kinetic energy of the center of mass.

Impulse and Momentum. Initially, the total momentum of the springbok is zero, but by the time it leaves the table it has a momentum that points up. This change in the total momentum of the system indicates that a net impulse is delivered to the springbok during phase 1. The normal force exerted by the table provides an upward impulse. The gravitational force due to Earth provides a downward impulse. The spring provides no net impulse, since it delivers an upward impulse to the upper mass and an equally large downward impulse to the lower mass. Of the normal force and the gravitational force, it is the normal force that provides the larger impulse. Students should wrestle with the fact that the spring does net work on the springbok, providing the springbok with its kinetic energy, but does not deliver the net impulse required to produce the springbok's upward momentum. Equally, students should appreciate that the normal force delivers a net impulse but does no work.

Using Pop-up Toys in the Classroom

We began our presentation with a quantitative analysis of a springbok. However, students can easily get lost in the details and learn very little about the underlying physics. To promote deeper learning, use a variety of approaches with your students.

You might want to start by asking students to *predict* which orientation of the springbok will lead to the highest jump, leaving open the possibility that the height of the jump is independent of the orientation. Discuss their reasoning and use some leading questions. For example, ask them to identify the objects that exert forces on the springbok and to describe the nature of these forces. Follow up by having them draw free-body diagrams. Ask them also to describe the energy in the springbok system, paying particular attention to how the energy gets redistributed as the system evolves in time. After the discussion, take a poll and then do a demonstration. Show connections with other "jumping" systems.

Working by *analogy* and determining the behavior of a system under a limited set of conditions is the stuff of working physicists. So, rather than focusing exclusively on the springbok, have students consider analogies. They could start by examining the design features of natural jumpers. How is the mass distributed among animals that jump? Can these systems jump effectively if turned upside down?

Another analogy is shown in Fig. 6, where two carts are launched from a wall using a compressed spring connecting them. When the heavier cart is away from the wall, the normal force is exerted over a longer time interval, making the impulse delivered to the system larger (see Fig. 7), which means that the center-of-mass velocity of the system is larger as well.

Student feeling of "cognitive overload" opens an opportunity to teach about the problem-solving process. Have your students *simplify the problem* until they obtain a version they can solve. Use motion diagrams and graphs. Having students sketch position versus time for different coordinates (e.g., the upper mass, lower mass, and center-of-mass coordinates) provides an excellent starting place for class-wide discussion. It also yields valuable information about students' conceptions of the motion. Or let them solve a series of related problems of increasing difficulty — (1) a cart connected to a spring launched from a wall, (2) a mass connected to a vertical spring launched from a table, (3) two carts connected to a spring launched from a wall, (4) a springbok, in the limit of a stiff spring, launched from a table.

Another good exercise is for students to consider possible *limiting cases*. An informative limit for the two-cart system is one in which one of the masses tends toward zero. In this limit, if the massive cart (see Fig. 6) is away from the wall, all of the elastic potential energy initially stored in the spring is transferred into kinetic energy associated with the translation of the center of mass. This leads to the maximum possible velocity for the center of mass. If the massive cart is against the wall, there is no way to transfer the elastic potential energy, which is stored in the (massless) spring, to the cart. The velocity of the center of mass will remain zero. For this limiting case, similar arguments can be made about the center-of-mass velocity of the springbok at the moment it loses contact with the table.

For a *hands-on project,* have students record the motion of a springbok with a video or digital camera. Save the recorded action to computer so students can analyze the positions of the individual masses and of the center of mass as functions of time. Students can use these data to check the model predictions (example at Ref. 1). Advanced students can explore more sophisticated models that take into account the mass of the spring and dissipative forces.

The great thing about a springbok is that different sets of concepts and activities can be used at different levels, and once the "pop-up" system is understood, students will be able to apply the lessons learned to real-life situations of bouncing balls, pogo sticks, or jumping animals.

References

1. http://umperg.physics.umass.edu/writings/springbok.
2. This assumes that there is enough potential energy stored in the spring to push the upper mass to height $D + d$, which happens only if $kD \geq (2m_u + m_l)g$.